Building a solid foundation from which to launch our future mathematicians

Robin Nagy
Cranbrook School, Sydney, NSW
<rnagy@cranbrook.nsw.edu.au>

It is essential to retain a focus on building students' mathematical reasoning and comprehension rather than merely developing superficial understanding through procedural learning. All too often this approach 'takes a back seat' because of examination and assessment pressure, where the importance of 'How?' supersedes that of 'Why?' It is not what we teach that is important so much as how we teach it. This session explores conceptual methods in the teaching of Secondary mathematics. It will appeal to both new and seasoned teachers, providing food for thought and suggesting practical approaches to teaching mathematics for understanding rather than regurgitation.

Introduction

Many teachers of mathematics find the time pressures and constraints of examination and assessment driving them towards teaching by rote learning instead of developing in their students a deep conceptual understanding of the material being covered. When students embark on university courses, their subsequent ability to cope with new material and novel problems and applications is hampered by their lack of solid mathematical foundations. Teachers need to be encouraged to present mathematics in a variety of ways which enhance the systemic understanding of concepts and the development of a systematic methodology.

Political and social pressures of examination-based assessment and achievement standards have inevitably dictated what and how we teach. The teaching of mathematics has, in many secondary school classrooms, become so dominated by assessment that 'the tail is wagging the dog'. A preoccupation exists to equip students with the skills necessary to 'pass the test' and this in turn, prescribes a procedural approach to teaching mathematics. This 'How' based style of teaching leads to concepts not being properly taught and understood, due to perceived time pressures on the teacher to get their students 'up to speed' on examination-style questions. Instead of examinations existing to assess mathematical knowledge and reasoning, they are seen as the *raison d'être* of the course and students are taught on a 'need to know' basis with exploration beyond the constraints of exam questions actively discouraged in many mathematics departments. Unfortunately, teaching 'to the test' is often an effective method of achieving good marks and it is possible to achieve creditable performances in mathematics examinations without really understanding any of the underlying concepts. By contrast, 'enrichment' is often seen as an intangible add-on for the brightest classes, an alternative to 'acceleration' and one which does not bring with it any concrete benefits.

In my opinion good teaching and good examination results are not mutually exclusive; indeed there is a strong positive correlation between the two. Mathematics teachers have an obligation to ensure that the 'Why' is taught together with the 'How' and that students' examination performance is indicative of their general comprehension of the subject. Learning is seldom a linear process and in order to develop mathematical

deductive reasoning, students will necessarily need to struggle to develop their own understanding and reasoning processes, with plenty of bumps and hiccups along the way. In the words of Lao Tsu (Giles, 1905, p. 45): "Failure is the foundation of success, and the means by which it is achieved."[1] This can be disconcerting for teachers and students alike and both must be prepared for a turbulent journey. Much of what is presented here could be labelled 'enrichment', which has often come to mean 'more than just teaching what they need to know (to answer questions)'. My firm contention is that the methodology of all teaching should endeavour to include such 'enrichment'.

Example topic 1: Pythagoras' Theorem

I have chosen to look at Pythagoras' Theorem by way of an example topic, to demonstrate the two different approaches to teaching mathematics, the procedural or 'How' and the conceptual or 'Why' approach.

A 'How' approach would involve teaching the formula $c^2 = a^2 + b^2$ for a right-angled triangle, explaining how to identify the hypotenuse and showcasing examples of typical questions which occur: finding the hypotenuse, finding one of the other sides, applying to 'real world' questions. This could be achieved in a few lessons with little or no conceptual enlightenment attained in the areas of mathematical proof or method, but rather a superficial understanding of how to answer questions based on a formula which we call Pythagoras' Theorem.

By contrast, a 'Why' approach might introduce the topic with a hands-on investigation such as Perigal's Dissection (Figure 1). Created in 1838 by Henry Perigal (1801–1898), a London Stockbroker and amateur mathematician, the construction consists of a right angled triangle with squares drawn on each of the sides. One of the adjacent sides is then dissected by drawing lines through its centre, parallel to the sides of the largest square and the four quadrilaterals formed can be rearranged, together with the square on the other adjacent side, to fit exactly inside the square on the hypotenuse. This is not a formal 'proof', but is a good graphical illustration and introduction to Pythagoras' Theorem; indeed Perigal (1891) postulated that Pythagoras probably discovered his theorem with a similar if not identical approach. The students can then be asked to propose a generalisation of their result using algebra. This is a constructivist approach to teaching Pythagoras' Theorem which can then be followed up by some examples demonstrating a more formal rigorous proof. It is a more powerful technique than the 'How' approach, as it encourages students to build their own knowledge and conclusions and to generalise their results. Not only will students be less likely to confuse the hypotenuse with the other two sides if they have this graphical foundation, but they will be more likely to remember and understand the theorem in the long-term.

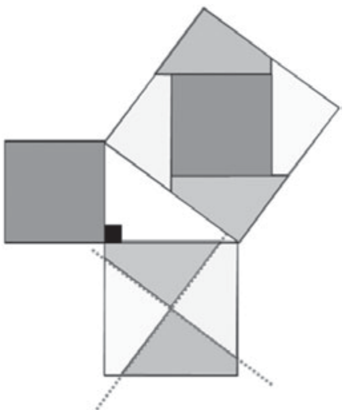

Figure 1. Perigal's dissection—a graphical illustration of Pythagoras' Theorem.

There are many proofs of Pythagoras' Theorem and students should be exposed to some of these, in order to understand the important mathematical concept of proof and its essential role in forming the structure of mathematical reasoning. Animated graphical proofs can be found on sites such as YouTube, for instance: http://www.youtube.com/watch?v=ajuUO8h0IxY and a variety of algebraic proofs are readily available from text books and online. Pythagoras' Theorem can then be applied to standard problems involving right-angled triangles with the students conceptually understanding the underlying 'truth' behind the theorem.

Example topic 2: The distance formula

Many teachers teach their students to find the distance between two points (x_1, y_1) and (x_2, y_2), using the 'distance formula':

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

But this formula could just as correctly be written:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1 Lao Tsu (c. 604–531 BC) was the founder of Taoism.

or $$d = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2}$$

This unwieldy formula simply represents Pythagoras' theorem where the two 'adjacent' sides are the difference in the x and y values of the coordinates. There are any number of such 'rules' to learn for a typical Senior Mathematics course and it is tempting to just tell students to 'learn' and apply the formulas by rote without attempting to explain where they come from.

A 'why' approach to teaching this topic emphasises that the distance between two points can be found by looking at the right-angled triangle formed by the difference in x and y coordinates (see Figure 2). The distance squared is the difference of the x coordinates squared plus the difference of the y coordinates squared. This provides a wonderful opportunity to introduce the symbols Δy and Δx meaning 'a change in' y and x respectively, long before calculus appears on the scene. It also provides a much more understandable formula: $d^2 = \Delta x^2 + \Delta y^2$ and facilitates the comprehension of the difference between, say, x values of 2 and -1 being 3 rather than 1 (these two values straddling the y-axis).

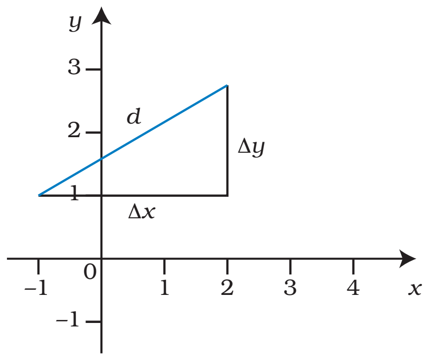

Figure 2. Considering Pythagoras' Theorem, the distance, d, between two points is the hypotenuse of a right-angled triangle, with adjacent sides formed by the difference in x and y values, Δx and Δy respectively. Thus: $d^2 = \Delta x^2 + \Delta y^2$.

Why do we learn mathematics? Where will I use this in life?

Perhaps the most haunting question in mathematics teachers' classrooms is: "Where is this going to be useful in life?" It is a common misconception that topics within mathematics are useful in most people's lives. Educators, politicians, text book and syllabus writers frequently fall into the trap of attempting to justify the teaching of mathematics by rationalising its usefulness to the real world. However, attempting to validate the place of mathematics in our curriculum merely on the grounds that it is 'useful' does the opposite. Excusing the learning of mathematics as merely being a 'useful' skill, minimises our discipline to one of utilitarianism. No other subject experiences such a pressure to validate its place in the classroom in terms of 'usefulness' to life. When do most people 'use' Art, English literature, music theory, history, or the sciences on a daily basis?

Whilst 'numeracy'[2] may indeed be useful, the sort of mathematical topics and procedures we teach from Year 7 onwards are not 'useful' in most people's day to day life. They can be invaluable in specific situations and specific occupations, but even as a teacher of mathematics, I do not 'use' simultaneous equations or trigonometry very often outside my teaching.

But that does not mean that it is not important to learn mathematics as a rigorous academic discipline. Mathematics is important. It is an abstract system of logical, deductive reasoning and methodology, which is pure and perfect (i.e., true). This discipline is useful, as it allows us to engage and communicate in higher order and abstract thinking across a spectrum of subjects and life events. Mathematics is the only thing we can 'prove' to be correct (based on some fundamental axioms). For most students, learning mathematics can be considered as a mental parallel to weight training. This analogy is a very effective way of explaining to students the importance of correct and systematic method and for addressing 'Why are we doing this?'-type questions. Of course, recent brain research highlighted in books such as *The Learning Revolution* by Dryden and Vos (1999) suggests that the process of studying (mathematics or otherwise), grows dendrites and makes connections between the neurons in the brain in an analogous way to weight training creating muscle fibre and toning the physique.

So the real benefit, to most students, of studying mathematics is that it develops their higher-order skills such as deductive reasoning and logical and organisational thinking. Once their brain has been developed in this way they *will* be able to use and further develop these 'brain muscles' in any number of useful contexts in their

2 In fact the word 'numeracy' is an ill-defined but ubiquitous term whose meaning appears to be commonly understood in politics and education to suit the given situation, but it was in fact invented by a committee in 1959 (the Crowther Report on UK Education) to represent the 'mirror image of literacy'.

lives, utilising pathways and connections originally developed in the mathematics classroom.

We can say that the 'effects of studying mathematics' are extremely useful for everybody!

Maths makes you 'mentally fit'

However, like weight training, brain development with mathematics only works effectively if you are doing the 'exercises' correctly. We like to find ways of doing exercises which do not 'hurt' and are easier, but they do not necessarily yield the same results. Another analogy I frequently use is to liken my role as a mathematics teacher to that of a personal trainer. A personal trainer may 'spot'[3] a client who is bench pressing, but that client will only be improving muscle tone if he himself is doing the majority of the work in moving the weights up and down (which involves intensive effort and focus). Once the personal trainer becomes the main source of power in moving the bar, the client is simply holding onto the bar whilst the personal trainer gives his own arms a good workout, lifting it up and down. From an observer's perspective the two situations appear identical, but only the former will yield muscle development in the client. It is easy to presume that just because students are answering mathematical questions, they are successfully learning, but I would contend that it is how they are learning which matters most in their brain development. Thus, the process is as all important in developing mathematical reasoning as it is with weight training in building and toning muscle fibre. These are all good analogies which can be used with students to convince them to set their work out correctly and take the time to work at a speed where they can be assured that each stage of their working is 100 per cent correct.

The method and process are the *only* things which matter in studying mathematics, not the answer—that is usually in the back of the book!

Example topic 3: Rearranging equations—a focus on method

Traditionally, rearranging equations was taught by learning the four 'rules' of taking items 'over the equals sign'. Adding became subtracting and vice versa and multiplication became division, but confusion often existed with which number was divided by which and why? Fortunately most teachers now use the analogy of a balance beam and explain that in order to maintain balance, whatever you do to one side of an equation you must do to the other. There are now many excellent animated visual resources, such as the 'Algebra Balance Scales' from the National Library of Virtual Manipulatives (Utah State University, 2010) which can be used to reinforce this analogous idea of an equation as a balance beam. This is a 'why' approach, but one which needs to be further enforced with a rigorous adherence to method.

I insist that my students write down at every stage what they are doing to both sides of an equation in manipulating the algebra (see Figure 3). This is the 'metacognition' of mathematics. I also insist on lining up the equals signs, so the correspondence to the fulcrum of a balance beam is maintained. I encourage students to use coloured pens and write down what they intend to do to both sides *before* writing the next line in their working. To this end, I always have a set of brightly coloured pens in my classroom and gladly give them out to students who want to use them (for the metacognition only!). It also helps if the ink smells of strawberries!

$$
\begin{aligned}
4x + 2 &= x + 8 && -2 \\
4x &= x + 6 && -x \\
3x &= 6 && \div 3 \\
x &= 2
\end{aligned}
$$

Figure 3. Setting out equations with the metacognition on the right hand side.

I emphasise to students that I am not particularly interested in the answers, which appear in the back of the book in any case—these are merely a way to check whether the working is error-free—I am only interested in the 'process instructions' which take you from line to line. I commend good working with a system of reward stamps and do not reward correct answers which are not set out in this exemplary manner. Initially students find the process monotonous, but so, I argue, is lifting weights—once they see how easily the result 'falls out', and how neat their work looks, they take it in their stride. I *always* set my work out like this on the board (at every level—including the highest level of Senior Mathematics—we must exemplify what we preach!).

3 'Spotting' in weight training means assisting in pushing a weight. Typically in order that the client maximises their physical capability to ensure optimal muscle development.

Inverse (or 'undoing') operations

I believe it to be important to introduce the concept of inverse (or 'undoing') processes as early as possible in Year 7 or 8. For instance, the inverse (or undoing function) of +3 is –3 (see Figure 4).

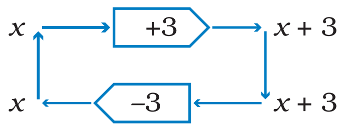

Figure 4. Function 'machines' showing the inverse processes +3 and –3.

If students are fluent with the concept of an inverse, it comes as no surprise that inverse trigonometry functions are required to 'undo' trigonometric functions. For instance, the inverse (undoing function) of sin( ) is called arcsin( ) or $\sin^{-1}(\ )$ (see Figure 5). I also like to use analogous 'real-world' functions and their inverses, such as silver plating and de-silver plating and discuss how different a function and its inverse are, typically, to each other. We also discuss 'self-inverse' functions such as the reciprocal function or, for instance the function $10–x$.

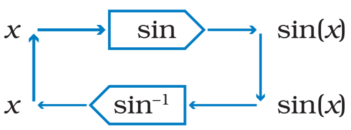

Figure 5. Function 'machines' showing the inverse processes sin( ) and sin–1( ).

We can see how to apply this to solve an equation $5=\dfrac{2}{\sin\theta}$ in Figure 6.

$$5=\frac{2}{\sin\theta}$$

$$\times\ \sin\theta$$

$$5\sin\theta=2$$

$$\div 5$$

$$\sin\theta=\frac{2}{5}$$

$$\sin^{-1}$$

$$\theta=\sin^{-1}\left(\frac{2}{5}\right)$$

$$\theta\approx 23.58°\,(4\text{sf})$$

Figure 6. Setting out correct metacognition (or 'what are you doing at every step to both sides') including an inverse trigonometric function to 'undo' the function sin( ).

Similarly, the inverse of an exponential function is called a logarithm (see Figure 7). This is how I introduce the topic of logarithms and builds a conceptual 'why' understanding rather than the more usual procedural definition of logarithms, adopted by most teachers and text books.

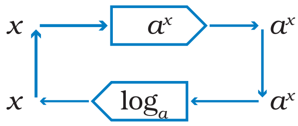

Figure 7. Function machines showing the inverse (undoing) of an exponential function a^x.

Logarithms can then be used to 'undo' their respective exponential functions as in Figure 8. Where the equation $2x + 3 = 91$ is solved in this manner.

$$2^{x+3}=91$$

$$\log_2(\ )$$

$$x+3=\log_2 91$$

$$-\ 3$$

$$x=(\log_2 91)-3$$

Figure 8. Using the base 2 logarithm function to 'undo' a base 2 exponential function 2^x.

Frustratingly, most calculators do not have a base n logarithm function, although this is beginning to change with new models[4]. Nevertheless, students can apply base 10 logarithms to generate a numerical answer to the same equation as can be seen in Figure 9, using logarithm rules. This generates a solution which is equivalent to that of Figure 8, and deduces the 'change of base' rule; we can conclude from this example that

$$\log_2 91=\frac{\log_{10}91}{\log_{10}2}$$

$$2^{x+3}=91$$

$$\log_{10}(\ )$$

$$\log_{10}2^{x+3}=\log_{10}91$$

$$(x+3)\log_{10}2=\log_{10}91$$

$$\div\ \log_{10}(2)$$

$$x+3=\frac{\log_{10}91}{\log_{10}2}$$

$$-\ 3$$

$$x=\frac{\log_{10}91}{\log_{10}2}-3$$

Figure 9. Using the base 10 logarithm function to 'undo' a base 2 exponential function 2^x.

4 It is alarming that some syllabuses do not allow these calculators to be used in their examinations as it does not allow meaningful assessment of the 'change of base' logarithm rule, which ironically has only come to prominence as a rule, due to the lack of a base n logarithm function on calculators; a perfect example of the 'tail wagging the dog'.

Conclusion

Mathematics is a science but mathematics teaching is an art; invention is the key to inspirational teaching and learning. It is important to keep fresh as a teacher of mathematics; to come up with new analogies and ways of explaining topics and never to be afraid to try something new and 'off-the-wall'. Sometimes it works and adds to your repertoire and sometimes it does not. But even if the analogy falls down or you realise you could have explained it better after you have already made an attempt, never forget that it is the process that is important—students will be learning from your mistakes as well as your polished set-pieces! Very often students will learn more from what goes wrong and how you (and they) work out what the problem is.

Education is all about the journey; this applies to the cyclical process of struggling, persisting and overcoming obstacles in producing new understanding and capacity for thought. It also applies equally to the outcome of education which hopefully remains with us long after we have left the classroom environment. American athlete Greg Anderson tells us to: "Focus on the journey, not the destination. Joy is found not in finishing an activity but in doing it." This is equally true of teaching and learning and is further epitomised by a quote from the tennis player Arthur Ashe: "Success is a journey, not a destination. The doing is often more important than the outcome." The word 'success' could easily be replaced with 'education'.

To use another of my analogies, there are several ways to guide a group through a forest:

- You can take them on well known tracks, enabling them to navigate the same track time and again, quickly and efficiently. This is not only boring, but virtually useless for future life unless they happen to be in the same forest on the same track.
- You can take them 'off-piste' using your own navigational skills. This will give them more of a sense of adventure and demonstrate that it is possible to reach the same destination in a variety of ways.
- You can teach them how to navigate themselves. You will be teaching them the skills which they can use time and again in their lives in many new and varied situations.

I enjoy inventing and taking new paths each time I teach, and believe that by doing so, I am equipping my students with the flexibility skills they need to be able to negotiate any new situation. Keeping notes for me is a sure way to become stale and I prefer to 'reinvent the wheel' with every new class, as an important strategy to keeping fresh, on my toes and exciting as a teacher. When a path leads nowhere, it is sometimes the best educational experience for your class (and perhaps for you too); in other words, when you make a mistake. You should never be frightened of this and indeed should emphasise it. You are a teacher, not a mathematics genius, but you might well be teaching one!

It is important to ask yourself 'Why?' when you teach every new concept and to avoid teaching 'recipes'. Let your students develop procedures of their own from the conceptual understanding they glean. If they are unable to do so, they may not have totally understood the concepts and could need more work on the foundations. Occasionally I feel compelled to teach 'recipes', especially where a syllabus appears to be specifically written to test recall of a formula rather than conceptual understanding of it; or when the constraints of departmental homogeneity dictate the length of time I am able to spend on a particular topic, but I always endeavour to explain to students my reasons for this and to revisit the topic at a later date if possible, for deeper conceptualisation.

The seventeenth century English statesman George Saville once remarked: "Education is what remains when we have forgotten all that we have been taught". I believe that the 'why' is the educational constituent of learning mathematics.

References

Dryden, G. & Vos, J. (1999). *The learning revolution* (2nd ed.). Auckland, NZ: The Learning Web Limited.

Giles, L. (1905). *The sayings of Lao-Tzu* (trans.). Retrieved 23 May 2013 from http://www.sacred-texts.com/tao/salt

Perigal, H. (1891). *Geometric dissections and transpositions.* London: Bell & Sons.

Utah State University. (2010). *National library of virtual manipulatives.* Retrieved 23 May 2013 from http://nlvm.usu.edu